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Nearly Vertical Hopf Bifurcation for a Passively Q-Switched Microchip Laser[1]

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Passively Q-switched microchip lasers generate strongly pulsating intensity oscillations that emerge from a Hopf bifurcation point. We show that this bifurcation is nearly vertical and explain why strongly pulsating oscillations are immediately observed as we pass the Hopf bifurcation point. The laser dynamical problem is mathematically a singular perturbation problem which we investigate. The leading order problem is conservative and corresponds to Lotka–Volterra equations.

KEY WORDS: Laser with a saturable absorber; passive Q-switching; singular Hopf bifurcation; Lotka–Volterra equations.

1. INTRODUCTION

Passively Q-switched lasers are lasers with an intracavity saturable absorber.[(1–2)] The absorption in the passive medium acts as an intensity dependent loss that changes the quality coefficient Q of the cavity. In case of strong saturability, the saturable absorber operates like a passive switch and, under some conditions, may switch spontaneously leading to short and intense pulses. These so-called passive Q-switching oscillations are of practical interest for applications that request extremely short (<1 ns) high-peak-power (>10 kW) pulses of laser light. The short pulse widths are useful for high-precision optical ranging with applications in automated production. The high peak output intensities are needed for efficient nonlinear frequency generation or ionization of materials, with applications in microsurgery and ionization spectroscopy. Previous numerical and analytical

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studies concentrated on passively Q-switched gas lasers.[4] They have shown that the passive Q-switching oscillations correspond to a limit-cycle which may emerge from a homoclinic orbit located very close to the laser threshold.[6, 7] This explains the pulsating nature and the relatively large period of these oscillations. Most theoretical studies have concentrated on gas lasers[4, 8] or semiconductor lasers.[5, 9] Recently, passively Q-switched solid state microchip lasers have been investigated both experimentally[10, 11] and numerically.[12, 13] By contrast to gas or semiconductor lasers, these microchip lasers generate strongly pulsating oscillations through a Hopf bifurcation mechanism. The main objective of this paper is to explain why these large amplitude oscillations are observed as soon as we pass the Hopf bifurcation point. As we shall demonstrate, the bifurcation is singular and needs to be analyzed in detail. Singular Hopf bifurcation problems have been studied in the past in order to describe the transition to pulsating chemical or biological oscillations.[20] However, the laser bifurcation is different and leads to more complex oscillations.

The paper is organized as follows. In Section 2, we introduce the rate equations for a large class of lasers. We show that these lasers naturally exhibit damped intensity oscillations which satisfy a degenerate form of Lotka–Volterra equations. In Section 3, we concentrate on passively Q-switched microchip lasers and investigate the Hopf bifurcation by using asymptotic methods. Lotka–Volterra equations appear as the leading order problem and allow us to describe the nearly vertical bifurcation branch.

2. SINGLE MODE LASER RATE EQUATIONS

A simple description of the process of amplification of light based on the ideas of spontaneous and stimulated emissions leads to a phenomenological description of the laser in terms of two equations for the atomic population inversion density and the electromagnetic energy density in the laser cavity (ref. 2, p. 13). In its simplest version, these rate equations apply to an idealized active system consisting of only two energy levels. If I represents the intensity of the laser field and if $N \equiv N_2 - N_1$ denotes the difference between the excited and ground state population densities, the dimensionless rate equations for I and N are given by

$$I' = I(-1 + N), \qquad N' = \gamma(A - N - NI) \tag{1}$$

In these equations, prime means differentiation with respect to time measured in units of κ^{-1}; κ is the decay rate of the field in the laser cavity due to loss of photons by mirror transmission, scattering, etc. A is the pump parameter which measures the energy injected into the laser. Its equals the population inversion in the absence of stimulated emission. If $A > 1$, a non zero intensity steady state is possible and represents the desired lasing action. $\gamma \equiv \gamma_{||}/\kappa$ is defined as the normalized decay rate of N towards its equilibrium state in the absence of laser light due to spontaneous emission. The following table gives typical values of κ and $\gamma_{||}$ for three different lasers used in laboratories and in applications.[14, 15] Although the values of κ and $\gamma_{||}$ are relatively different between lasers, we note that γ is typically an $O(10^{-3})$ small quantity. For microchip solid state lasers, γ is even smaller. This small value of γ is responsible for the laser intensity oscillations as we shall now explain.

laser	κ (s^{-1})	$\gamma_{\|\|}$ (s^{-1})
CO_2	10^8	2.5×10^5
solid state	10^6	4×10^3
semiconductor laser	10^{12}	10^9

If $A > 1$, the non-zero intensity solution is the only stable long time regime. However, it is well know that the laser may exhibit transient intensity oscillations if the steady state is perturbed. These oscillations are called the laser relaxation oscillations and can be analyzed by investigating the limit $\gamma \to 0$ of Eq. (1).[16–18] Assuming $N - 1 = O(\sqrt{\gamma})$, this limit leads to the equations of a conservative oscillator given by

$$I' = I(-1 + N), \qquad N' = \gamma(A - 1 - I) \tag{2}$$

These equations admit a one parameter family of periodic solutions which are useful when we analyze the effect of a small external perturbation such as a periodic modulation of a parameter, an injected signal, or optical feedback. Equation (2) only admits the non zero intensity steady state (the zero intensity steady state has moved to infinity in the limit $\gamma \to 0$). The possible similitude between Eq. (2) and the Lotka–Volterra equations which exhibit both a center and a saddle has been suspected (first reference in ref. 17) and discussed in connection with the two mode laser problem.[19] Indeed, Eq. (2) is equivalent to a degenerate form of Lotka–Volterra equations (see Appendix). In the next section, we show that the complete Lotka–Volterra equations appear as the leading order equations for passively Q-switched microchip lasers.

3. PASSIVE Q-SWITCHED MICROCHIP LASERS

It was recently shown that passively Q-switched microchip lasers are capable of producing short and intense pulses.[10, 11] The small size of these lasers along with their simplicity make them attractive for many applications. This has motivated the recent interest to model and simulate these pulses.[12, 13] The pulsating response of microchip lasers is well described by rate equations. Assuming a fast absorber, Eq. (1) is modified as

$$I' = I\left(-1 + N - \frac{\bar{A}}{1+\alpha I}\right) \tag{3}$$

$$N' = \gamma(A - N - NI) \tag{4}$$

In these equations, $\bar{A}$ is the pump parameter of the absorbing medium. The parameter α represents the relative saturability of the absorber with respect to the amplifying medium. Estimations of the parameters for microchip lasers[12, 13] indicates that γ is extremely small ($\gamma = O(10^{-6}$–$10^{-7})$), that $\alpha < 1$ ($\alpha = O(10^{-1}$–$10^{-2})$) and that $\bar{A} = O(1)$ is either larger or smaller than one depending on the absorber. The control parameter $A = O(1)$ is assumed larger but close to the laser first threshold defined by

$$A_{\mathrm{th}} \equiv 1 + \bar{A} \tag{5}$$

As A is progressively increased, the zero intensity solutions looses its stability at $A = A_{\mathrm{th}}$. A non zero intensity steady state then appears and is given by

$$I \simeq \frac{A - A_{\mathrm{th}}}{A_{\mathrm{th}} - \alpha \bar{A}} \tag{6}$$

for A close to A_{th}. If α is sufficiently small, $A_{\mathrm{th}} - \alpha\bar{A}$ is positive and the bifurcation is supercritical and stable. This is the case for the microchip lasers[13] but not for gas or semiconductor lasers which exhibit an α much larger than 1. As A is further increased, the stable steady state solution (6) undergoes a Hopf bifurcation at a critical intensity $I = I_H$. From the linear stability analysis, we find that I_H satisfies the condition

$$\bar{A}\alpha I - \gamma(1+I)(1+\alpha I)^2 = 0 \tag{7}$$

and approaches the limit

$$I_H \simeq \frac{\gamma}{\bar{A}\alpha} \tag{8}$$

as $\gamma \to 0$. The corresponding values of $A = A_H$ and $N = N_H$ are obtained from the steady state equations evaluated at $I = I_H$ i.e.,

$$N_H = 1 + \frac{\bar{A}}{1 + \alpha I_H} \qquad \text{and} \qquad A_H = N_H(1 + I_H) \tag{9}$$

Numerical studies of the periodic solution of Eqs. (3) and (4) for progressively smaller values of γ indicates that the limit-cycle solution

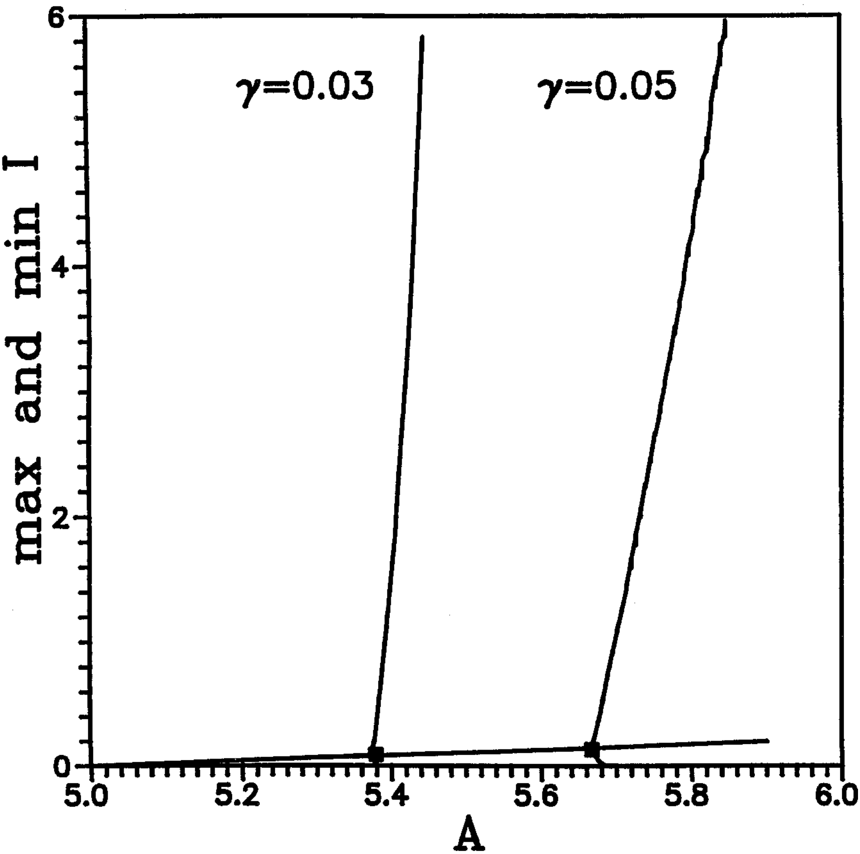

Fig. 1. Hopf bifurcation to pulsating intensity oscillations. The bifurcation diagram of the limit-cycle solutions of Eqs. (3) and (4) is determined for two different values of γ. The values of the fixed parameters are $\bar{A} = 4$ and $\alpha = 0.1$. The figure represents the maximum and minimum of the intensity $I(t)$ as a function of the control parameter A. The figure also shows the nonzero intensity steady state that appears at $A_{\text{th}} = 5$. Note that the branch of periodic solutions moves to A_{th} as we decrease γ and becomes more and more vertical: for $\gamma = 0.05$, the change of the maximum intensity occurs in a small domain $A - A_H \sim 0.18$ while for $\gamma = 0.03$, the same change appears in the domain $A - A_H \sim 0.06$.

quickly becomes pulsating as we deviate from the Hopf bifurcation point. Figure 1 shows the bifurcation diagram of the periodic solutions for two different values of γ. The intensity oscillations are nearly harmonic as $A - A_H$ increases from zero but become strongly pulsating as soon as the minimum intensity approaches zero. The limit-cycle oscillations then correspond to a sequence of high intensity pulses separated by long intervals where the intensity is almost zero.

We wish to investigate the behavior of the Hopf bifurcation branch as $\gamma \to 0$. To this end, we apply asymptotic methods appropriate for the description of singular Hopf bifurcations.[20] We first rewrite Eqs. (3) and (4) in terms of deviations from the Hopf bifurcation point. Specifically, we introduce the new variables s, x and y defined by

$$s = \gamma t, \qquad N = N_H + \gamma x, \qquad I = I_H(1 + y) \tag{10}$$

and expand $A - A_H$ in power series of γ as

$$A - A_H = \gamma\lambda_1 + \gamma^2\lambda_2 + \cdots \tag{11}$$

Introducing (10) and (11) into Eqs. (3) and (4) leads to the following equations for x and y

$$x' = -x(1 + I_H) - \gamma^{-1} N_H I_H y - I_H xy + (\lambda_1 + \gamma\lambda_2 + \cdots) \tag{12}$$

$$y' = (1 + y)\left[x + \frac{\bar{A} a I_H \gamma^{-1} y}{(1 + \alpha I_H)(1 + \alpha I_H(1 + y))}\right] \tag{13}$$

where prime means differentiation with respect to s. Using (8) and the fact that $N_H = A_{\text{th}} + O(\gamma)$, the leading order problem as $\gamma \to 0$ is given by

$$x' = -x - \frac{A_{\text{th}}}{\bar{A} a} y + \lambda_1 \tag{14}$$

$$y' = (1 + y)(x + y) \tag{15}$$

The right hand side of Eq. (15) suggests that we may further simplify these equations if we introduce the new dependant variables

$$u = x + y \qquad \text{and} \qquad v = y \tag{16}$$

In terms of (16), Eqs. (14) and (15) become

$$u' = v(-\omega^2 + u) + \lambda_1 \tag{17}$$

$$v' = (1 + v)\, u \tag{18}$$

where ω is defined by

$$\omega \equiv \sqrt{\frac{A_{\text{th}}}{\bar{A}\alpha} - 1} \tag{19}$$

We note that Eqs. (17) and (18) are Lotka–Volterra equations if $\lambda_1 = 0$ (see Appendix). These equations admit an one-parameter family of periodic

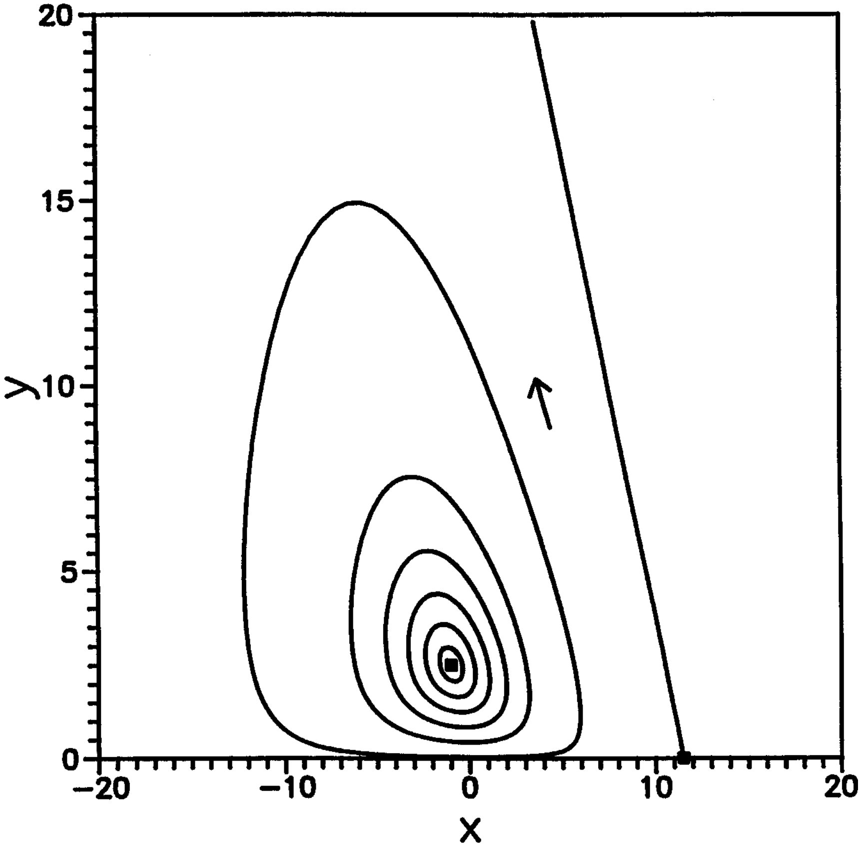

Fig. 2. Periodic solutions. The Lotka–Volterra equations are conservative and admit an one-parameter family of periodic solutions. We represent six orbits in the phase plane (x, y). x and y are the physical variables and represent the deviations of N and I from their Hopf bifurcation values (see (10)). The orbits are surrounding a center and are bounded by two separatrices which emerge from a saddle point (the singular points and the separatrices are also shown in the figure). The arrow indicates the direction of the trajectories. The different orbits have been obtained from Eqs. (14) and (15). The values of the parameters are $\bar{A} = 4$ and $\alpha = 0.1$.

solutions (see Fig. 2). The oscillations are nearly harmonic and $2\pi/\omega$-periodic close to the center $(u, v) = 0$ and become pulsating as the maximum intensity (max of v) increases. If $\lambda_1 \gtrless 0$, Eqs. (17) and (18) admit either decaying or growing oscillations. Thus, the condition for bounded periodic solutions requires that $\lambda_1 = 0$. We conclude that the bifurcation branch is vertical at this order of the perturbation analysis. The next correction of the bifurcation parameter (i.e., λ_2) appears at the next order of the perturbation analysis and a solvability condition allows us to describe how the amplitude of (u, v) changes in the thin domain $A - A_H = O(\gamma^2)$. The formulation of the bifurcation equation is not difficult but its solution needs to be determined numerically. The details of this study do not contribute to our physical understanding of the laser sharp bifurcation and will be given elsewhere.

4. SUMMARY

The laser intensity oscillations are intimately connected to Lotka–Volterra oscillations. In the first part of our analysis, we showed that the laser nearly conservative oscillations (or laser relaxation oscillations) satisfy a degenerate form of the Lotka–Volterra equations. In the second part of our analysis, we considered a microchip solid state laser with a saturable absorber which admits a Hopf bifurcation to strongly pulsating oscillations. The limiting problem is now given by the complete Lotka–Volterra equations. This observation has two immediate consequences on our physical understanding of the laser oscillations. First, Lotka–Volterra equations describe how the limit-cycle oscillations deform as we increase the amplitude along the nearly vertical bifurcation. Second, the Lotka–Volterra equations admit two simple separatrices (namely, $v = -1$ and $u = \omega^2$) which are connecting a saddle point. As the amplitude of the periodic solution becomes large, the limit-cycle orbit becomes asymmetric and spend most of its time near these separatrices. This suggests to reexamine the limit-cycle solution by determining separate approximations valid for different parts of the orbit. We may benefit from earlier asymptotic studies of the Lotka–Volterra large amplitude oscillations.[(23)]

APPENDIX. LOTKA–VOLTERRA EQUATIONS

Volterra's (1926) interest[(21)] arose from an observed ecological situation involving two interacting populations. The model equations he suggested are the same as those proposed by Lotka (1920)[(22)] for a hypothetical reaction mechanism. The Lotka–Volterra equations for U and V are given by

$$U' = U(a - bV), \qquad V' = V(-c + dU) \tag{20}$$

where the coefficients a, b, c and d are positive constants.

We first show that the reduced laser rate equations (2) are equivalent to a degenerate form of Eq. (20). Consider the limit $b \to 0$ of Eq. (20) assuming

$$a = O(b), \qquad c = O(b^{-p}) \qquad \text{and} \qquad d = O(1) \tag{21}$$

where $0 < p < 1$. This limit means that the non-zero steady state of Eq. (20), given by

$$(U_s, V_s) = \left(\frac{c}{d}, \frac{a}{b}\right) \tag{22}$$

moves to infinity as $b \to 0$ (more precisely, $U_s \to \infty$ but V_s remains fixed). Introducing the new variables

$$W = d\left(U - \frac{c}{d}\right), \qquad V = v \tag{23}$$

and assuming $W = O(1)$, Eq. (20) simplifies as

$$W' = b(c + W)\left(\frac{a}{b} - V\right) \simeq bc\left(\frac{a}{b} - V\right) \tag{24}$$

$$V' = WV \tag{25}$$

as $b \to 0$. The system of Eqs. (24) and (25) is identical to Eq. (2) with $W = N - 1$, $V = I$, $bc = \gamma = O(b^{1-p})$ and $a/b = A - 1 = O(1)$.

We next show that Eqs. (17) and (18) with $\lambda_1 = 0$ are equivalent to Lotka–Volterra equations. Substituting

$$u = \omega^2 - V, \qquad v = -1 + U \tag{26}$$

into Eqs. (17) and (18), we obtain Eq. (20) with $b = c = d = 1$ and $a = \omega^2$.

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